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Phase space models for stochastic nonlinear parabolic waves: wave spread and collapse

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Abstract

We derive several kinetic equations to model the large scale, low Fresnel number behaviour of the generalized nonlinear Schrödinger (NLS) equation with a rapidly fluctuating random potential. Depending on the relative scale of fluctuation in the longitudinal and transverse directions, we classify the kinetic equations into the longitudinal, the transverse and the isotropic case. The principal assumption of our derivation is that the rapid fluctuation in the linear potential does *not* give rise to rapid oscillations in the *modulus* of the wave amplitude. For the longitudinal and the transverse cases we address two problems, the rate of dispersion and the singularity formation, using the nonlinear kinetic equations. The main technique is the variance identities derived for the nonlinear kinetic equations. For the problem of dispersion, we show that in the longitudinal case the spread scales like $(\text{time})^{3/2}$ whereas in the transverse case the spread is linearly proportional to time. For the problem of singularity formation, we show that the collapse conditions in the transverse case remain the same as those for the homogeneous NLS equation with critical or supercritical self-focusing nonlinearity whereas in the longitudinal case the small-scale medium fluctuations tend to enhance the energy of the system and thus raise the energy barrier to wave collapse.

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1. Introduction

The cubic nonlinear Schrödinger (NLS) equation is a prototypical nonlinear wave equation arising in diverse fields from nonlinear fibre optics to plasma physics to the Bose–Einstein condensation and many others [18]. In this paper, we consider the generalized NLS equation with a random potential

$$\mathrm{i}\frac{\partial}{\partial z}\Psi(z,\mathbf{x})+\frac{\gamma}{2}\Delta_{\mathbf{x}}\Psi(z,\mathbf{x})+\gamma^{-1}g|\Psi|^{2\sigma}\Psi(z,\mathbf{x})+\mu V(zL_z,\mathbf{x}L_x)\Psi(z,\mathbf{x})=0,\qquad \mathbf{x}\in\mathbb{R}^d,\quad \sigma>0 \tag{1}$$

where $\gamma = L_z/(kL_x^2)$ is the Fresnel number (k is the carrier wavenumber), μ is the linear coupling coefficient for the random potential V, which is rescaled by two large parameters L_z and L_x, and g is the nonlinear coupling coefficient with $g > 0$ representing the self-focusing (attractive) case and $g < 0$ is the self-defocusing (repulsive) case. Here σ is a positive constant and $\sigma = 1$ corresponds to the cubic NLS equation. We are particularly interested in the regime of low Fresnel number $\gamma \ll 1$ with a rapidly oscillating potential $L_x, L_z, \mu \gg 1$. The timelike variable z is the longitudinal coordinate in the direction of wave propagation and we will refer to it as 'time' in what follows while we will refer to the transverse coordinates $\mathbf{x} \in \mathbb{R}^d$ as the space variables. We shall assume the random potential V is a z-stationary, $\mathbf{x}$-homogeneous Gaussian process whose two-point correlation, and hence the probability distribution, is uniquely determined by its power spectral density Φ as in

$$\mathbb{E}[V(z,\mathbf{x})V(z',\mathbf{x}')] = \int \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\mathrm{e}^{\mathrm{i}\xi(z-z')}\Phi(\xi,\mathbf{k})\,\mathrm{d}\xi\,\mathrm{d}\mathbf{k},$$

where $\mathbb{E}$ stands for the expectation.

Finite-time singularity or wave collapse is a well-known effect for the self-focusing, (super)-critical ($d\sigma \geqslant 2$) NLS equation without a random potential [18] when the nonlinear focusing effect dominates over the linear diffraction effect. Recently it has been proved that for a white-noise-in-z potential ($L_z \to \infty$, L_x fixed) the solution of NLS equation can still develop singularity in finite time (see [8] for an elementary proof). Indeed, without the self-averaging effect of an $\mathbf{x}$-rapidly fluctuating potential the large z-fluctuations in the white-noise potential may drive the system to a state with low, negative Hamiltonian, thus developing singularities in finite time. The presence of an $\mathbf{x}$-rapidly fluctuating potential, however, may still be able to delay or prevent wave collapse or singularity formation. Given the wide range of scales present in such a question numerical simulation as well as theoretical analysis are undoubtedly extremely challenging.

In this paper, we consider the generalized NLS with a random potential fluctuating *rapidly but on different scales* in the longitudinal and transverse directions and propose several phase-space model equations corresponding to different longitudinal and transverse scalings. The main assumption of our derivation is the *absence* of small-scale fluctuations in the *modulus* $|\Psi|$ of the wave amplitude in the scaling limits. In other words, we assume that the rapid fluctuations of the random potential give rise to rapid fluctuations *only* in the *phase* of the wave amplitude. We then use these kinetic equations to elucidate the problems of dispersion and singularity formation. The main ingredient of our analysis is the variance identities and energy law for these kinetic equations which are of independent interest.

The variance identities for the longitudinal case derived in this paper include the *ensemble-averaged* variance identities derived in [8] where the white-noise-in-z potential was considered (see the comments following (39)–(40)). This means that the second moment in $\mathbf{x}$ of the corresponding kinetic model derived here describes the ensemble-averaged second moment in $\mathbf{x}$ of the white-noise-in-z case and provides an indirect evidence for the validity of our assumption of the absence of small-scale fluctuations in $|\Psi|$. It is an open question if the same correspondence exists between the transverse case and the white-noise-in-$\mathbf{x}$ case.

We summarize our findings as follows. When the random medium fluctuates much faster in the transverse direction (the transverse case), the dispersion rate in position is linear in z for the critical or defocusing nonlinearity. With the focusing interaction the linear-in-z

behaviour is only an upper and lower bound for the supercritical and subcritical nonlinearity, respectively. On the other hand, when the random medium fluctuates much faster in the longitudinal direction (the longitudinal case), the dispersion rate in position is $O(z^{3/2})$ for the critical or defocusing nonlinearity. With the focusing interaction the $z^{3/2}$ behaviour is only an upper and lower bound for the supercritical and subcritical nonlinearity, respectively.

As a consequence, the small-scale random scattering in the transverse case does not affect the classical wave-collapse condition in the homogeneous case whereas in the longitudinal case it tends to raise the energy barrier to collapse.

2. Wigner distribution and Wigner–Moyal equation

Our phase-space model equations for the low Fresnel number regime are based on the Wigner equations. The Wigner distribution for pure state Ψ is defined as

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{y}}\Psi\left(\mathbf{x}+\frac{\gamma\mathbf{y}}{2}\right)\Psi^*\left(\mathbf{x}-\frac{\gamma\mathbf{y}}{2}\right)\mathrm{d}\mathbf{y} \tag{2}$$

from which the wave amplitude Ψ can be recovered up to a constant phase factor by using

$$\Psi(\mathbf{x}_1)\Psi^*(\mathbf{x}_2) = \int W\left(\frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2),\mathbf{q}\right)\exp[i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)/\gamma]\,\mathrm{d}\mathbf{q}.$$

The Wigner distribution has many useful properties. For instance, partial integration of W gives rise to the marginal distributions

$$\int W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{p} = |\Psi(\mathbf{x})|^2 \qquad \int W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x} = \left(\frac{2\pi}{\gamma}\right)^d\left|\hat{\Psi}\left(\frac{\mathbf{p}}{\gamma}\right)\right|^2.$$

More generally, integration of W on any hyperplane in the phase space is related to the squared modulus of the fractional Fourier transform of Ψ.

Consequently, the mean $\bar{\mathbf{x}}$ and variance V_x of $\mathbf{x}$ are given by, respectively,

$$\begin{aligned}\bar{\mathbf{x}} &= \int \mathbf{x}W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p},\\ V_x &= \int |\mathbf{x}-\bar{\mathbf{x}}|^2 W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}\\ &= S_x - \bar{\mathbf{x}}^2\end{aligned}$$

where S_x is the second moment of $\mathbf{x}$

$$S_x = \int |\mathbf{x}|^2 W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}.$$

The spatial variance V_x measures the dispersion of the wave about its centre of mass $\bar{\mathbf{x}}$.

Also, we have the identity for the flux density

$$\frac{1}{2\mathrm{i}}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) = \int_{\mathbb{R}^d}\mathbf{p}W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{p} \tag{3}$$

so that the expression

$$\bar{\mathbf{p}} = \int \mathbf{p}W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}$$

has the meaning of average momentum. The momentum variance V_p is then defined as

$$V_p = \int |\mathbf{p}-\bar{\mathbf{p}}|^2 W(\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} = S_p - \bar{\mathbf{p}}^2$$

where S_p is the second moment of $\mathbf{p}$

$$S_p = \int |\mathbf{p}|^2 W(\mathbf{x}, \mathbf{p})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{p}.$$

The momentum variance V_p measures the dispersion of the momentum about its average $\bar{\mathbf{p}}$.

In view of these properties it is tempting to think of the Wigner distribution as a phase-space probability density, which is unfortunately not the case, since it is not everywhere non-negative (it is always real-valued though).

It is straightforward to derive the closed-form equation for the Wigner distribution [7]

$$\frac{\partial W}{\partial z} + \mathbf{p} \cdot \nabla_{\mathbf{x}} W + \mathcal{U}_\gamma W + \mathcal{V} W = 0, \tag{4}$$

with the Moyal operators

$$\mathcal{U}_\gamma W(\mathbf{x}, \mathbf{p}) = \mathrm{i} \int \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \gamma^{-1} [W(\mathbf{x}, \mathbf{p} + \gamma\mathbf{q}/2) - W(\mathbf{x}, \mathbf{p} - \gamma\mathbf{q}/2)]\, \widehat{U}(z, \mathbf{q})\, \mathrm{d}\mathbf{q}, \qquad U = g|\Psi|^{2\sigma} \tag{5}$$

$$\mathcal{V} W(z, \mathbf{x}, \mathbf{p}) = \mathrm{i}\mu \int \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}L_x} [W_z(\mathbf{x}, \mathbf{p} + \theta\mathbf{q}/2) - W_z(\mathbf{x}, \mathbf{p} - \theta\mathbf{q}/2)]\, \widehat{V}(zL_z, \mathrm{d}\mathbf{q}), \tag{6}$$

where $\hat{U}$ is the Fourier transform of U and $\hat{V}$ is the (transverse) spectral measure of the z-stationary, $\mathbf{x}$-homogeneous random field V. $\hat{V}$ is related to the (transverse) power spectral density Φ_0 as follows:

$$\mathbb{E}[\hat{V}(z, \mathrm{d}\mathbf{p})\hat{V}(z, \mathrm{d}\mathbf{q})] = \delta(\mathbf{p} + \mathbf{q})\Phi_0(\mathbf{p})\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{q}.$$

The transverse power spectrum density in turn is related to the full power spectrum density $\Phi(w, \mathbf{p})$ in the following way:

$$\Phi_0(\mathbf{p}) = \int \Phi(w, \mathbf{p})\, \mathrm{d}w.$$

Without loss of generality we may assume

$$\Phi(w, \mathbf{p}) = \Phi(-w, \mathbf{p}) = \Phi(w, -\mathbf{p}) = \Phi(-w, -\mathbf{p}). \tag{7}$$

Note that the operator $\mathcal{U}_\gamma$ has the following formal geometrical optics limit as $\gamma \to 0$:

$$\mathcal{U}_\gamma W(\mathbf{x}, \mathbf{p}) \longrightarrow \mathcal{U}_0 W(z, \mathbf{x}, \mathbf{p}) \equiv \nabla_{\mathbf{x}} U \cdot \nabla_{\mathbf{p}} W(\mathbf{x}, \mathbf{p})$$

would be important for the derivation of the kinetic equations later.

One advantage of working with equation (4) is that one can use it to evolve the mixed-state initial condition, instead of the pure-state one given in (2). This is important in the context of modelling quantum open systems. The mixed-state Wigner distribution is a convex combination of the pure-state Wigner distributions (2). The main advantage in the present context, however, lies in the relative ease of dealing with any low Fresnel number behaviour $\gamma \to 0$.

Let $\{\Psi_\alpha\}$ be a family of L^2 functions parametrized by α which is weighted by a probability measure $P(\mathrm{d}\alpha)$. Denote the pure-state Wigner distribution (4) by $W[\Psi]$. A mixed-state Wigner distribution is given by

$$\int W[\Psi_\alpha] P(\mathrm{d}\alpha). \tag{8}$$

The limits as $\gamma \to 0$ of the mixed state Wigner distributions constitute the so-called Wigner measures which are always positive [11, 12, 14]. Evolution by equation (4) preserves the form (8). In particular, for such initial data we have

$$\int W(z, \mathbf{x}, \mathbf{p})\, \mathrm{d}\mathbf{p} \geqslant 0, \qquad \forall\, \mathbf{x} \in \mathbb{R}^d \tag{9}$$

$$\int W(z, \mathbf{x}, \mathbf{p})\,\mathrm{d}\mathbf{x} \geqslant 0, \qquad \forall\, \mathbf{p} \in \mathbb{R}^d. \tag{10}$$

Multiplying (4) by W and integrating by parts we also see that the evolution preserves the $L^2(\mathbb{R}^{2d})$-norm of W, i.e.

$$\frac{\partial}{\partial z}\int |W|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} = 0.$$

3. Radiative transfer limits and kinetic equations

We consider the whole family of scaling limits, called the radiative transfer limits, which are roughly defined as $\gamma \to 0$, $L_z, L_x \to \infty$, and further distinguished by whether

$$\theta \equiv \lim_{\gamma\to 0} \gamma L_x = 0 \quad \text{or a positive number} \tag{11}$$

as well as the relative size of L_z and L_x. This, of course, is not sufficient to ensure the existence of scaling limit until we specify the strength of V and its mixing property.

3.1. Linear kinetic equations.

First we summarize what has been established in the linear case when $U(z, \mathbf{x})$ is a *given* function. In this case we refer to equation (4) with a given U as the linear Wigner–Moyal (LWM) equation. The principal feature of the scaling is the *separation of scales* in the given potential U and the random potential V.

Under the integrability of the maximum correlation coefficient of $V(z, \cdot)$ as $\mathbf{x}$-homogeneous-field-valued z-stationary process (among other minor conditions) we proved in [7, 9] that with the scaling limit (11) the weak solution of the LWN equation converges *in probability* to the weak solution of the linear Boltzmann (LB) equation or the linear Fokker–Planck (LFP) equation, described below, depending on whether θ also tends to zero or not.

Linear Boltzmann equation ($\theta = 1$):

$$\frac{\partial}{\partial z}W(z, \mathbf{x}, \mathbf{p}) + \mathbf{p}\cdot\nabla_{\mathbf{x}}W(z, \mathbf{x}, \mathbf{p}) + \nabla_{\mathbf{x}}U(z, \mathbf{x})\cdot\nabla_{\mathbf{p}}W(z, \mathbf{x}, \mathbf{p})$$
$$= 2\pi \int K(\mathbf{p}, \mathbf{q})[W(z, \mathbf{x}, \mathbf{q}) - W(z, \mathbf{x}, \mathbf{p})]\,\mathrm{d}\mathbf{q} \tag{12}$$

with a nonnegative kernel $K(\mathbf{p}, \mathbf{q})$ given respectively as follows.

(a) If $\mu \sim \sqrt{L_z}$, $L_x \ll L_z$ then

$$K(\mathbf{p}, \mathbf{q}) = \Phi(0, \mathbf{q} - \mathbf{p}). \tag{13}$$

(b) If $\mu \sim \sqrt{L_x}$, $L_z \ll L_x \ll L_z^{4/3}$, $d \geqslant 3$ then

$$K(\mathbf{p}, \mathbf{q}) = \delta\left(\frac{|\mathbf{q}|^2 - |\mathbf{p}|^2}{2}\right)\left[\int \Phi(w, \mathbf{q} - \mathbf{p})\,\mathrm{d}w\right]. \tag{14}$$

(c) If $\mu \sim \sqrt{L_z}$, $L_x \sim L_z$ then

$$K(\mathbf{p}, \mathbf{q}) = \Phi\left(\frac{|\mathbf{q}|^2 - |\mathbf{p}|^2}{2}, \mathbf{q} - \mathbf{p}\right). \tag{15}$$

Linear Fokker–Planck equation ($\theta \to 0$):

$$\frac{\partial}{\partial z} W(z, \mathbf{x}, \mathbf{p}) + \mathbf{p} \cdot \nabla_{\mathbf{x}} W(z, \mathbf{x}, \mathbf{p}) + \nabla_{\mathbf{x}} U(z, \mathbf{x}) \cdot \nabla_{\mathbf{p}} W(z, \mathbf{x}, \mathbf{p}) = \nabla_{\mathbf{p}} \cdot \mathbf{D} \nabla_{\mathbf{p}} W(z, \mathbf{x}, \mathbf{p}) \tag{16}$$

with a symmetric, nonnegative-definite matrix $\mathbf{D}$ given as follows.

(a) If $\mu \sim \theta^{-1}\sqrt{L_z}$, $L_x \ll L_z$ then

$$\mathbf{D} = \pi \int \Phi(0, \mathbf{q}) \mathbf{q} \otimes \mathbf{q} \, \mathrm{d}\mathbf{q}. \tag{17}$$

(b) If $\mu \sim \theta^{-1}\sqrt{L_x}$, $L_z \ll L_x \ll L_z^{4/3}$, $d \geqslant 3$ then

$$\mathbf{D}(\mathbf{p}) = \pi |\mathbf{p}|^{-1} \int_{\mathbf{p}\cdot\mathbf{p}_\perp = 0} \int \Phi(w, \mathbf{p}_\perp) \, \mathrm{d}w \mathbf{p}_\perp \otimes \mathbf{p}_\perp \, \mathrm{d}\mathbf{p}_\perp. \tag{18}$$

(c) If $\mu \sim \theta^{-1}\sqrt{L_z}$, $L_x \sim L_z$ then

$$\mathbf{D}(\mathbf{p}) = \pi \int \Phi(\mathbf{p} \cdot \mathbf{q}, \mathbf{q}) \mathbf{q} \otimes \mathbf{q} \, \mathrm{d}\mathbf{q}. \tag{19}$$

We shall use $\mathcal{L}$ to denote either the scattering operator on the right-hand side of equation (12) or the diffusion operator on the right-hand side of equation (16). The self-adjoint operator $\mathcal{L}$ is non-positive definite and represents various decoherence effects due to random fluctuations of the medium; see (21). Let us point out one striking feature of the above family of scaling limits: the passage is from random, unitary evolution governed by (4) to deterministic, L^2-contracting evolution governed by (12) or (16). This relies on the weak formulation of (4) which results in self-averaging and irreversibility in the scaling limits.

We refer to regime (a), for either (12) or (16), as the *longitudinal* case and regime (b) as the *transverse* case. Regime (c) is the isotropic, borderline case. We will consider hereafter only the longitudinal and transverse cases which have a clear-cut dispersion rate.

We note that the restrictions of $L_x \ll L_z^{4/3}$ and $d \geqslant 3$ in regime (b) are due to technical reasons and we believe that they can be lifted off (see [7, 9] for details). Previously it was shown in [6, 17] (see also [16]) that for a Gaussian potential with $\theta = 1$, $L_z = 0$ and $d \geqslant 3$, the mean field $\mathbb{E}W$, converges to (12) (b).

3.2. Nonlinear kinetic equation.

When $U = g|\Psi|^{2\sigma}$, the convergence of the above scaling limits is not known. The missing link, at least at the conceptual level, is the separation of scales of U and V which requires to show that the fast-scale oscillation of Ψ due to V is present only in the the phase, as V is real-valued, and hence disappears in U. In other words, we assume that U converges *strongly* in the limit. The strong convergence of $\rho = |\Psi|^2$ has been proved for the *linear* longitudinal case [15].

Below we shall postulate this scenario of separation of scales and the validity of the above scaling limits in the nonlinear case with the self-interaction potential $U = g|\Psi|^{2\sigma}$. We will use the resulting nonlinear kinetic equation (12) or (16) to investigate nonlinear wave spread and singularity in the presence of various random potentials of different scalings. Such a model with the kernel similar to (14) was considered in [10].

Let us state the nonlinear kinetic equation which we will analyse subsequently:

$$\frac{\partial}{\partial z} W(z, \mathbf{x}, \mathbf{p}) + \mathbf{p} \cdot \nabla_{\mathbf{x}} W(z, \mathbf{x}, \mathbf{p}) + \mathcal{U}_0 W(z, \mathbf{x}, \mathbf{p}) = \mathcal{L} W(z, \mathbf{x}, \mathbf{p}) \tag{20}$$

where $\mathcal{U}_0 = \nabla_{\mathbf{x}} U \cdot \nabla_{\mathbf{p}}$ and $\mathcal{L}$ is either the linear scattering operator

$$\mathcal{L}W = 2\pi \int K(\mathbf{p},\mathbf{q})[W(z,\mathbf{x},\mathbf{q}) - W(z,\mathbf{x},\mathbf{p})]\,\mathrm{d}\mathbf{q}$$

or the linear diffusion operator

$$\mathcal{L}W = \nabla_{\mathbf{p}} \cdot \mathbf{D}\nabla_{\mathbf{p}} W(z,\mathbf{x},\mathbf{p}).$$

The nonlinear kinetic equation (20) preserves the total mass, i.e.

$$\frac{\partial}{\partial z} N = 0, \qquad N = \int W(z,\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}$$

but in general decreases the L^2-norm

$$\frac{\partial}{\partial z} \int |W|^2(z,\mathbf{x},\mathbf{p})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} = \int W\mathcal{L}W\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} \leqslant 0 \tag{21}$$

because the operator $\mathcal{L}$ is non-positive definite. Equation (21) is a form of Boltzmann's H-theorem. The inequality (21) expresses certain irreversibility as a result of the weak convergence of solutions [7]. One can absorb the effect of the total mass N into g by the obvious rescaling of W in equation (20). Henceforth we assume that

$$N = 1.$$

3.3. Initial condition

A natural space of initial data and solutions is the space $\mathcal{S}$ of the non-negative measures with square integrable density W

$$\int |W|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} < \infty,$$

finite Dirichlet form

$$-\int W\mathcal{L}W\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} < \infty$$

and finite, positive variances V_x, V_p as well as $S_x, S_p \in (0,\infty)$. In addition, we shall also assume the initial data to have a finite Hamiltonian H

$$H = \frac{1}{2}S_p - \frac{g}{\sigma+1}\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x} \in (-\infty,\infty), \qquad \rho(\mathbf{x}) = \int W\,\mathrm{d}\mathbf{p}.$$

The first term of the Hamiltonian is the kinetic energy and the second term is the self-interaction potential energy. A finite H and a finite S_p imply a finite self-interaction potential

$$\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x} < \infty. \tag{22}$$

Let $[0, z_*)$ be the maximal interval on which the regular solution is defined with the above properties. When $z_* < \infty$ then the solution is said to develop finite time singularity.

3.4. Energy law

Next, we consider the evolution of the Hamiltonian H and the variance V_x. Let us first note the result of $\mathcal{U}_\gamma$ applied to the quadratic polynomials.

$$\mathcal{U}_\gamma \mathbf{x} = 0 \tag{23}$$

$$\mathcal{U}_\gamma \mathbf{p} = \mathrm{i}\int \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{q}\hat{U}(\mathrm{d}\mathbf{q}) = \nabla_{\mathbf{x}} U \tag{24}$$

$$\mathcal{U}_\gamma |\mathbf{x}|^2 = 0 \tag{25}$$

$$\mathcal{U}_\gamma \mathbf{x}\cdot\mathbf{p} = \mathrm{i}\int \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\mathbf{x}\cdot\mathbf{q}\hat{U}(\mathrm{d}\mathbf{q}) = \mathbf{x}\cdot\nabla_{\mathbf{x}} U \tag{26}$$

$$\mathcal{U}_\gamma |\mathbf{p}|^2 = \mathrm{i}\int \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} 2\mathbf{p}\cdot\mathbf{q}\hat{U}(\mathrm{d}\mathbf{q}) = 2\mathbf{p}\cdot\nabla_{\mathbf{x}} U. \tag{27}$$

It is noteworthy that the results of the calculation are independent of $\gamma \geqslant 0$ and identical to those for $\gamma = 0$ (see more on this in the conclusion).

Consider the mean dynamics for

$$\bar{\mathbf{x}} = \int \mathbf{x} W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}, \qquad \bar{\mathbf{p}} = \int \mathbf{p} W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} \tag{28}$$

with the mean Hamiltonian defined as

$$\bar{H} = \tfrac{1}{2}|\bar{\mathbf{p}}|^2. \tag{29}$$

Using the above and integrating by parts we obtain the following,

$$\frac{\partial}{\partial z}\bar{\mathbf{p}} = \int \mathcal{L}\mathbf{p} W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}$$

$$\frac{\partial}{\partial z} V_p = \int \nabla_{\mathbf{x}} U \cdot \mathbf{p} W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} + \int \mathcal{L}|\mathbf{p}|^2 W \,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{x}$$

$$\frac{\partial}{\partial z}\frac{g}{\sigma+1}\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x} = \int \nabla_{\mathbf{x}} U \cdot \mathbf{p} W d\mathbf{p}\,\mathrm{d}\mathbf{x}$$

and hence

$$\frac{\partial}{\partial z}\bar{H} = \bar{\mathbf{p}}\cdot\int \mathcal{L}\mathbf{p} W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} \tag{30}$$

$$\frac{\partial}{\partial z} H = \int \mathcal{L}|\mathbf{p}|^2 W \,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{x} \tag{31}$$

which will take a more explicit form once we calculate $\mathcal{L}\mathbf{p}$, $\mathcal{L}|\mathbf{p}|^2$ with $\mathcal{L}$ of each case.

In the following sections, we first derive the variance identities for equation (20) in the longitudinal case (regime (a)) and then the transverse case (regime (b)).

Before ending this section let us state some elementary inequalities which will be useful later. An application of the Cauchy–Schwartz inequality and the marginal positivity (9), (10) to the first moments $\bar{\mathbf{x}}$, $\bar{\mathbf{p}}$ then leads to

$$|\bar{\mathbf{x}}|^2 \leqslant S_x, \qquad |\bar{\mathbf{p}}|^2 \leqslant S_p. \tag{32}$$

Furthermore, by using the mixed-state structure (8) in estimating the cross moment

$$S_{xp} = \int \mathbf{x}\cdot\mathbf{p} W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} \tag{33}$$

one deduces that

$$S_{xp}^2 \leqslant S_x S_p < \infty. \tag{34}$$

Likewise the covariance

$$V_{xp} = \int (\mathbf{x}-\bar{\mathbf{x}})\cdot(\mathbf{p}-\bar{\mathbf{p}}) W \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p} = S_{xp} - \bar{\mathbf{x}}\cdot\bar{\mathbf{p}}$$

can be bounded by V_x and V_p as

$$V_{xp}^2 \leqslant V_x V_p. \tag{35}$$

4. The longitudinal case

4.1. Variance identities

Here we extend the classical variance identities (Virial theorem) to the phase-space models (20).

We have the following simple calculations:

$$\begin{aligned}
\mathcal{L}\mathbf{x} &= 0 \\
\mathcal{L}\mathbf{p} &= 2\pi \int \Phi(0, \mathbf{p}-\mathbf{q})(\mathbf{p}-\mathbf{q})\,\mathrm{d}\mathbf{q} = 0 \quad \text{(by (7))} \\
\mathcal{L}|\mathbf{x}|^2 &= 0 \\
\mathcal{L}\mathbf{x}\cdot\mathbf{p} &= \mathbf{x}\cdot 2\pi \int \Phi(0, \mathbf{p}-\mathbf{q})(\mathbf{p}-\mathbf{q})\,\mathrm{d}\mathbf{q} = 0 \quad \text{(by (7))} \\
\mathcal{L}|\mathbf{p}|^2 &= -2\pi \int \Phi(0, \mathbf{p}-\mathbf{q})\left[|\mathbf{p}|^2 - |\mathbf{q}|^2\right]\mathrm{d}\mathbf{q} \\
&= -2\pi \int \Phi(0, \mathbf{q})(2\mathbf{p}-\mathbf{q})\cdot\mathbf{q}\,\mathrm{d}\mathbf{q} \\
&= 2\pi \int \Phi(0, \mathbf{q})|\mathbf{q}|^2\,\mathrm{d}\mathbf{q} \equiv R.
\end{aligned} \tag{36}$$

We have used the linear Boltzmann operator $\mathcal{L}$ in the above calculation; the same result holds for the linear diffusion operator $\mathcal{L}$ for which case

$$R = 2 \times \mathrm{trace}(\mathbf{D}),$$

where the diffusion matrix $\mathbf{D}$ is given by (17). We shall use the above identities to perform integrating by parts in the derivation of the variance identities.

The evolution of the mean position $\bar{\mathbf{x}}$ and momentum $\bar{\mathbf{p}}$ is then given by

$$\frac{\partial}{\partial z}\bar{\mathbf{x}} = \bar{\mathbf{p}} \qquad \frac{\partial}{\partial z}\bar{\mathbf{p}} = \frac{g\sigma}{\sigma+1}\int \nabla_{\mathbf{x}}\rho^{\sigma+1}\,\mathrm{d}\mathbf{x} = 0$$

as a consequence the mean Hamiltonian $\bar{H}$ is invariant.

The evolution of the variance V_x is given by

$$\frac{\partial}{\partial z}V_x = 2V_{xp}.$$

Differentiating S_{xp} we obtain

$$\frac{\partial}{\partial z}S_{xp} = S_p - \frac{g\,\mathrm{d}\sigma}{\sigma+1}\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}$$

Hence the second derivative of V_x becomes

$$\frac{\partial^2}{\partial z^2}V_x = 4(H-\bar{H}) + \frac{2(2-\mathrm{d}\sigma)g}{\sigma+1}\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x}. \tag{37}$$

An alternative expression for the variance identity is

$$\frac{\partial^2}{\partial z^2}V_x = 2\,\mathrm{d}\sigma(H-\bar{H}) + (2-\mathrm{d}\sigma)V_p. \tag{38}$$

Both forms (37) and (38) will be used to obtain dispersion estimates below.

A slightly different version of the variance identities can be analogously derived:

$$\frac{\partial^2}{\partial z^2}S_x = 4H + \frac{2(2-\mathrm{d}\sigma)g}{\sigma+1}\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x}. \tag{39}$$

$$\frac{\partial^2}{\partial z^2} S_x = 2\,\mathrm{d}\sigma H + (2 - \mathrm{d}\sigma) S_p. \tag{40}$$

As we will see below, both (37)–(38) and (39)–(40) hold for the transverse case as well as the longitudinal case. What is interesting about (39)–(40) is that they coincide with the ensemble-averaged variance identities derived rigorously in [8] for the case of the white-noise-in-z potential. On the other hand, (37)–(38) cannot be true for the white-noise (in z or $\mathbf{x}$) case even after ensemble-averaging because $\mathbb{E}|\bar{\mathbf{x}}|^2$ and $\mathbb{E}|\bar{\mathbf{p}}|^2$ involve the second moment of W. In the following we will use (37)–(38) exclusively since they give a slightly better estimate for the dispersion rate than (39)–(40).

4.2. Dispersion rate

Although the medium is lossless, reflected in the fact that the total mass $N = 1$ is conserved, but the Hamiltonian is not conserved by the evolution since the scattering with the random potential is not elastic. Indeed, its rate of change is

$$\frac{\partial}{\partial z} H = R, \qquad H(z) = H(0) + Rz \tag{41}$$

due to the diffusion-like spread in the momentum $\mathbf{p}$.

In the critical case $\mathrm{d}\sigma = 2$, we have the exact result

$$\frac{\partial^2}{\partial z^2} V_x = 4H - 2|\bar{\mathbf{p}}|^2 = 4(H(0) - \bar{H}) + 4Rz$$

before any singularity formation and hence the following.

Proposition 1. *If* $\mathrm{d}\sigma = 2$ *or* $g = 0$, *then*

$$V_x(z) = V_x(0) + 2V_{xp}(0)z + 2(H(0) - \bar{H})z^2 + \frac{2R}{3} z^3, \qquad z \in [0, z_*). \tag{42}$$

The analogous result ($V_x \sim z^3$) for the *linear* Schrödinger equation ($d = 1, g = 0$) with a random potential has been proved previously [2, 5].

We have from (37) that

$$\frac{\partial^2}{\partial z^2} V_x = 4H + \frac{(4 - 2\,\mathrm{d}\sigma)g}{\sigma + 1} \int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x} - 2|\bar{\mathbf{p}}|^2 \tag{43}$$

and hence

$$\frac{\partial^2}{\partial z^2} V_x \leqslant 4H - 4\bar{H} = 4(H(0) + Rz) - 4\bar{H}, \qquad \text{for} \quad g(2 - \mathrm{d}\sigma) < 0$$

$$\frac{\partial^2}{\partial z^2} V_x \geqslant 4H - 4\bar{H} = 4(H(0) + Rz) - 4\bar{H}, \qquad \text{for} \quad g(2 - \mathrm{d}\sigma) \geqslant 0.$$

On the other hand, from (38) we obtain for any g

$$\frac{\partial^2}{\partial z^2} V_x \leqslant 2\,\mathrm{d}\sigma (H - \bar{H}), \qquad \text{for} \quad 2 - \mathrm{d}\sigma \leqslant 0$$

$$\frac{\partial^2}{\partial z^2} V_x \geqslant 2\,\mathrm{d}\sigma (H - \bar{H}), \qquad \text{for} \quad 2 - \mathrm{d}\sigma \geqslant 0.$$

Integrating the above inequalities twice, we obtain the following.

Proposition 2 (attractive interaction)**.** *For $g \geqslant 0$, we have*

$$V_x(z) \leqslant V_x(0) + 2V_{xp}(0)z + \mathrm{d}\sigma(H(0) - \bar{H})z^2 + \frac{\mathrm{d}\sigma}{3}Rz^3, \qquad \text{for} \quad 2 \leqslant \mathrm{d}\sigma$$

$$V_x(z) \geqslant V_x(0) + 2V_{xp}(0)z + \mathrm{d}\sigma(H(0) - \bar{H})z^2 + \frac{\mathrm{d}\sigma}{3}Rz^3, \qquad \text{for} \quad 2 \geqslant \mathrm{d}\sigma$$

for all $z \in [0, z_*)$.

Proposition 3 (repulsive interaction)**.** *Assume $g < 0$ (hence $H \geqslant 0$). Then*

$$V_x(z) \leqslant V_x(0) + 2V_{xp}(0)z + (\mathrm{d}\sigma \vee 2)[H(0) - \bar{H}]z^2 + \frac{\mathrm{d}\sigma \vee 2}{3}Rz^3$$

and

$$V_x(z) \geqslant V_x(0) + 2V_{xp}(0)z + (\mathrm{d}\sigma \wedge 2)[H(0) - \bar{H}]z^2 + \frac{\mathrm{d}\sigma \wedge 2}{3}Rz^3 \tag{44}$$

for $z \in [0, z_*)$.

4.3. Singularity formation

Finite-time singularity for the critical or supercritical NLS ($\mathrm{d}\sigma \geqslant 2$) in the absence of an external potential is a well-known effect [18]. In this case the singularity is the blow-up type $V_p, \|\rho\|_{\sigma+1} \to \infty$. Here we take (20) as a model equation to gain some insight into singularity formation in the presence of a rapidly fluctuating random potential.

First we consider the self-focusing case $g \geqslant 0$. For $\mathrm{d}\sigma \geqslant 2$ one can bound V_x as

$$V_x(z) \leqslant V_x(0) + 2V_{xp}(0)z + \mathrm{d}\sigma(H(0) - \bar{H})z^2 + \frac{\mathrm{d}\sigma R}{3}z^3 \equiv F(z) \tag{45}$$

and looks for the situation when $F(z)$ vanishes.

The sufficient conditions for $F(z)$ to vanish at a finite positive z are that $F(z)$ takes a non-positive value $F(z_0) \leqslant 0$ at its local minimum point $z_0 > 0$. The local minimum point z_0 is given by

$$z_0 = \frac{\bar{H} - H(0) + \sqrt{(H(0) - \bar{H})^2 - 2RV_{xp}(0)/(\mathrm{d}\sigma)}}{R}. \tag{46}$$

Therefore we are led to the singularity conditions for $g \geqslant 0$.

Proposition 4. *For $\mathrm{d}\sigma \geqslant 2$, $g > 0$, the regular solution of the nonlinear kinetic equation (20) with the longitudinal randomness develops singularity at a finite time $z_* \leqslant z_0$ given by (46) under the condition $F(z_0) \leqslant 0$ and either one of the following conditions:*

$$V_{xp}(0) < 0 \tag{47}$$

$$V_{xp}(0) > 0, \qquad H(0) < \bar{H} - \sqrt{\frac{2RV_{xp}(0)}{\mathrm{d}\sigma}}. \tag{48}$$

Remark 1. Clearly, the condition $F(z_0) \leqslant 0$ requires $H(0)$ to be sufficiently below $\bar{H}$ by allowing the potential energy

$$-\frac{g}{\sigma+1}\int \rho^{\sigma+1}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{p}$$

at $z = 0$ to be sufficiently negative.

In the linear or self-defocusing case $g \leqslant 0$ the right-hand side of (42) or (44) can be shown to be always positive by using the inequality

$$2|V_{xp}| \leqslant V_x + V_p,$$

cf (35).

Condition (48) suggests that the small-scale random fluctuations raise the energy barrier for singularity formation since $H(0) < \bar{H}$ leads to finite-time singularity in the absence of random potential. This is different from the result of [3] for white-noise-in-z potentials which are smooth and *slowly* fluctuating in $\mathbf{x}$. Such random potentials tend to trigger wave collapse.

Next we will follow the argument of [11] to show more explicitly the blow-up mechanism in the case with the supercritical, self-focusing nonlinearity and give a sharper bound on z_* under certain circumstances.

Proposition 5. *Suppose* $\mathrm{d}\sigma > 2$, $g > 0$. *Then under conditions (47),*

$$H(z_0) = H(0) + Rz_0 \leqslant 0 \tag{49}$$

with z_0 *given by (46),* V_x' *as well as* V_p *blow up at a finite time* z_* *where*

$$z_* \leqslant z_0 \wedge \frac{2V_x(0)}{V_{xp}(0)(2 - \mathrm{d}\sigma)}.$$

Let us give the argument below. Since blow-up is a local phenomenon, V_x is a poor indicator of its occurrence. To this end a more useful object to consider is V_p.

From (38) it follows that

$$\frac{\partial^2}{\partial z^2} V_x \leqslant (2 - \mathrm{d}\sigma) V_p < 0, \qquad z < z_*. \tag{50}$$

Hence $V_x'(z)$ is a negative, decreasing function for $z < z_*$. Also by the Cauchy-Schwartz inequality, we get

$$0 \leqslant V_{xp}^2 \leqslant V_x V_p \leqslant V_x(0) V_p, \qquad 0 < z < z_* \tag{51}$$

and hence

$$V_p \geqslant \frac{V_{xp}^2}{V_x(0)}. \tag{52}$$

Let $A(z) = -V_x'(z) > 0$, $z < z_*$. We have from (38) and (51) the differential inequality

$$\frac{\partial}{\partial z} A \geqslant CA^2, \qquad C = \frac{\mathrm{d}\sigma - 2}{4V_x(0)} > 0 \tag{53}$$

which yields the estimate

$$A(z) \geqslant \frac{A(0)}{1 - zCA(0)}, \qquad z < \frac{1}{CA(0)}$$

and thus the blow-up of $-V_x'(z)$ at a finite time. This along with (52) then (49) implies the divergence of V_p at a finite time

$$\lim_{z \to z_*} V_p(z) = \infty$$

with

$$z_* \leqslant \frac{1}{CA(0)} = \frac{2V_x(0)}{V_{xp}(0)(2 - \mathrm{d}\sigma)}. \tag{54}$$

For high power $\mathrm{d}\sigma \gg 2$, (54) is a better upper bound for z_* than z_0 given by (46).

The preceding argument demonstrates clearly the blow-up mechanism, namely the quadratic growth property (53) as well as giving a sharper bound on the blow-up time for large $d\sigma$.

5. The transverse case

The scattering operator $\mathcal{L}$ in this case corresponds to elastic scattering, instead of the inelastic scattering of the longitudinal case. This affects the results of $\mathcal{L}$ when applied to quadratic polynomials:

$$\mathcal{L}\mathbf{x} = 0 \tag{55}$$

$$\mathcal{L}\mathbf{p} = 2\pi \int \delta\left(\frac{|\mathbf{p}|^2 - |\mathbf{q}|^2}{2}\right)\left(\int \Phi(w, \mathbf{p}-\mathbf{q})\,\mathrm{d}w\right)(\mathbf{p}-\mathbf{q})\,\mathrm{d}\mathbf{q} = 0 \tag{56}$$

$$\mathcal{L}|\mathbf{x}|^2 = 0 \tag{57}$$

$$\mathcal{L}\mathbf{x}\cdot\mathbf{p} = 0 \tag{58}$$

$$\mathcal{L}|\mathbf{p}|^2 = -2\pi \int \delta\left(\frac{|\mathbf{p}|^2 - |\mathbf{q}|^2}{2}\right)\left(\int \Phi(w, \mathbf{p}-\mathbf{q})\,\mathrm{d}w\right)\left(|\mathbf{p}|^2 - |\mathbf{q}|^2\right)\mathrm{d}\mathbf{q} = 0. \tag{59}$$

The same results can be easily checked to hold for the diffusion operator $\mathcal{L}$ with the diffusion matrix (18). The main difference between the transverse and the longitudinal cases is that between (59) and (36).

5.1. Rate of dispersion

The variance identities (37)–(38) and (39)–(40) still hold in the transverse case as can be derived by using (55)–(59) and other relations as in the longitudinal case. The difference from the longitudinal case is that the Hamiltonian is invariant

$$\frac{\partial}{\partial z}H = 0$$

due to (59).

By the same argument as in the longitudinal case we have the following analogous estimates.

Proposition 6. *If* $d\sigma = 2$ *or* $g = 0$, *then*

$$V_x(z) = V_x(0) + 2V_{xp}(0)z + 2(H(0) - \bar{H})z^2, \qquad z \in [0, z_*). \tag{60}$$

Proposition 7 (attractive interaction)**.** *The following estimates hold for* $g \geqslant 0$ *and* $z \in [0, z_*)$*:*

$$V_x(z) \geqslant V_x(0) + 2V_{xp}(0)z + d\sigma(H - \bar{H})z^2, \qquad \text{for} \quad 2 \geqslant d\sigma$$

$$V_x(z) \leqslant V_x(0) + 2V_{xp}(0)z + d\sigma(H - \bar{H})z^2, \qquad \text{for} \quad 2 \leqslant d\sigma.$$

Proposition 8 (repulsive interaction)**.** *Assume* $g < 0$ *(hence* $H \geqslant 0$*). Then for* $z \in [0, z_*)$

$$V_x(0) + 2V_{xp}(0)z + (d\sigma \wedge 2)[H - \bar{H}]z^2 \leqslant V_x(z) \leqslant V_x(0) + 2V_{xp}(0)z + (d\sigma \vee 2)[H - \bar{H}]z^2. \tag{61}$$

From these estimates we see that a ballistic kind of motion takes place in the transverse case.

5.2. Singularity

By finding the zeros of the upper bounds in proposition 7 we can derive the conditions for singularity formation. As in the longitudinal case, for $g \leqslant 0$, (60) and the left-hand side of (61) are always positive.

Proposition 9. *For $g > 0, \mathrm{d}\sigma \geqslant 2$, the regular solution of the nonlinear kinetic equation (20) with the transverse randomness develops singularity in a finite time $z_* < \infty$ under either of the following conditions*

$$H < \bar{H} \tag{62}$$

or

$$V_{xp}(0) < 0, \qquad \bar{H} \leqslant H \leqslant \bar{H} + |V_{xp}(0)|^2/(\mathrm{d}\sigma V_x(0)). \tag{63}$$

Remark 2. It is easy to see that the singularity time z_* is bounded from above by

$$z_0 = \frac{-V_{xp}(0) + \sqrt{|V_{xp}(0)|^2 - \mathrm{d}\sigma V_x(0)(H - \bar{H})}}{\mathrm{d}\sigma(H - \bar{H})} \tag{64}$$

under condition (62) and by

$$z_0 = \frac{-V_{xp}(0) - \sqrt{|V_{xp}(0)|^2 - \mathrm{d}\sigma V_x(0)(H - \bar{H})}}{\mathrm{d}\sigma(H - \bar{H})}$$

under condition (63).

The preceding singularity conditions are identical to those for the homogeneous (super)-critical NLS equation [18]. This is already suggested by the previous result [10] where (62) is shown to be the instability condition for the diffusion approximation of the kinetic equation (20) with the kernel (14).

With more stringent conditions one can demonstrate more explicitly the phenomena of wave collapse. Let us state the result.

Proposition 10. *Suppose* $\mathrm{d}\sigma > 2$, $g > 0$. *Then under the conditions*

$$H < \bar{H},$$

V_x' as well as V_p blow up at a finite time $z_* < z_0$.

Let us sketch the argument below. From (38) it follows that

$$\frac{\partial^2}{\partial z^2} V_x \leqslant (2 - \mathrm{d}\sigma) V_p < 0, \qquad z < z_*. \tag{65}$$

Since $V_x'(z)$ is a decreasing function and becomes negative after a finite time $\bar{z}$ when $V_x(z)$ achieves its maximum $\bar{V}_x$.

Again for $A(z) = -V_x'(z) > 0, z > \bar{z}$ we have from (65) and (51) the differential inequality

$$\frac{\partial}{\partial z} A \geqslant C A^2, \qquad C = \frac{\mathrm{d}\sigma - 2}{4 V_x(\bar{z})} > 0, \qquad z > \bar{z}.$$

6. Conclusion

We have derived several kinetic equations to model the large scale, low Fresnel number behaviour of the generalized NLS equation with a rapidly fluctuating random potential based on the rigorous theory [7, 9] for the linear case. This is the so-called radiative transfer theory. The main hypothesis is that the small-scale fluctuations induced by the random medium shows up only in the phase of Ψ but not in the modulus $|\Psi|$, namely $\rho = |\Psi|^2$ converges *strongly* in the scaling limit. As a consequence, the low Fresnel number waves interacting with a rapidly oscillating potential give rise to, in the radiative transfer scaling (11), a self-averaging limit of a deterministic kinetic equation with a scattering operator. The scattering operator in the kinetic equation may be a Boltzmann-type operator or a Fokker–Planck operator depending on how small the Fresnel number is. From these kinetic equations we have derived the variance identities in order to shed light on two problems: the rate of dispersion and the singularity formation.

What is important for our investigation is the *structure* of the scattering kernel in the Boltzmann operator or the diffusion matrix of the Fokker–Planck operator. We have considered two types of structures: the longitudinal case when the random potential fluctuates more rapidly in the longitudinal direction and the transverse case when the random potential fluctuates more rapidly in the transverse direction.

For the problem of dispersion, roughly speaking, we have shown that in the longitudinal case the spread scales like $z^{3/2}$, whereas in the transverse case the spread scales like z.

For the problem of singularity, we have shown by analysing the variance identities that in the longitudinal case the random scattering tends to boost the energy of the system and therefore raise the energy barrier to singularity whereas in the transverse case the singularity conditions remain the same as those for the homogeneous NLS equation with critical or supercritical self-focusing nonlinearity.

Finally, the variance identities (39)–(40) can be rigorously derived for the case of the white-noise-in-z potential after ensemble averaging, [8]. This is consistent with the assumed self-averaging property of the radiative transfer scaling limit. It would be of great interest to know if the variance identities (39)–(40) for the transverse case can be derived for the case of the white-noise-in-$\mathbf{x}$ potential.

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